

On thin-shell wormholes evolving in flat FRW spacetimes

M. La Camera*

Department of Physics and INFN - University of Genoa

Via Dodecaneso 33, 16146 Genoa, Italy

We analyze the stability of a class of thin-shell wormholes with spherical symmetry evolving in flat FRW spacetimes. The wormholes considered here are supported at the throat by a perfect fluid with equation of state $\mathcal{P} = w\sigma$ and have a physical radius equal to aR , where a is a time-dependent function describing the dynamics of the throat and R is the background scale factor. The study of wormhole stability is done by means of the stability analysis of dynamic systems.

PACS numbers: 04.20.Cv, 04.20.Gz

Keywords: Thin-shell wormholes; dynamic systems

1. Introduction

Wormhole physics has become a popular research topic since the classical paper by Morris and Thorne.¹ Early work was reviewed in the book of Visser² and there is an extensive recent review by Lobo.³ The class of thin-shell wormholes provides a wide collection of examples for the study of traversable Lorentzian wormholes. Visser^{4,5} studied wormholes in Schwarzschild and Reissner-Nordström backgrounds using for the first time the cut-and-past technique. The stability of static and dynamic wormholes has been examined either by choosing specific equations of state or by considering a linearized stability analysis around a static solution. The literature is extremely extensive but for some examples see Refs. 6-19 and references therein. In this work we analyze the stability of a class of thin-shell wormholes with spherical symmetry evolving in a flat Friedmann-Robertson-Walker (FRW) cosmological background. Dynamic Lorentzian wormholes connecting FRW spacetimes were first studied by Hochberg and Kephart⁷ who discussed a possible resolution of the horizon problem using a network of evolving wormholes present in the early universe. We utilize the

*Electronic address: lacamera@ge.infn.it

wormhole equations of motion given in Ref. 7 and assume that the matter at the shell is a perfect fluid with equation of state $\mathcal{P} = w\sigma$. We shall obtain an evolution equation for the physical radius aR of the wormhole, where a is a time-dependent function describing the dynamics of the throat and R is the background scale factor. The study of stability is done considering the wormhole physical radii as dynamic systems and analyzing the behavior of their steady states with respect to a set of control parameters affecting the evolution.

2. Thin-shell wormholes in cosmology

The dynamic wormholes we shall consider here result from surgically grafting two FRW spacetimes as proposed by Hochberg and Kephart ⁷ following the method used by Visser ^{4,5} for constructing wormholes by cutting and pasting two manifolds to form a geodesically complete new manifold with a shell placed in the joining surface. The construction of Ref. 7, which we now briefly recall, starts taking two copies of the FRW solution

$$dS^2 = R^2(t) \left(\frac{dr^2}{1 - k r^2} + r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) \right) - dt^2 \quad (1)$$

and then removes from every one of them an identical four-dimensional region of the form $\Sigma_{\pm} = \{r_{\pm} \leq a\}$ where $a = a(t)$ is a time-dependent radius. The resulting manifold contains two disjoint boundaries $\Sigma_{\pm} = \{r_{\pm} = a\}$ and by the identification $\Sigma_+ = \Sigma_- \equiv \Sigma$ one obtains two FRW spacetimes connected by a traversable wormhole with spherical symmetry whose throat is located on their mutual boundary $\Sigma = \{f(r, t) = r - a(t) = 0\}$. The corresponding geometry can be analyzed using the Israel-Lanczos-Sen ^{20–22} “thin-shell” formalism of General Relativity. The wormhole throat is a timelike hypersurface with interior coordinates $\xi^i = (\vartheta, \varphi, \tau)$, τ being the throat proper time. The position of the throat in the background FRW spacetime is given by $X^\mu = (a(t), \vartheta, \varphi, t)$. The line element (1) with the curvature term k neglected is well justified to describe not only, as in Ref. 7, the early stages of expansion but also the universe today on large scales, so we shall put $k = 0$ in it. Moreover another realization of an evolving wormhole was obtained by imagining its asymptotically parts as constituting flat FRW spacetimes. ⁹ The tangent vectors to Σ have the components $e_{(i)}^\mu|_{\pm} = \partial X_{\pm}^\mu / \partial \xi^i$ and the induced metric $g_{ij} = e_{(i)}^\mu e_{(j)}^\nu g_{\mu\nu}|_{\pm}$ on the junction hypersurface gives the line element

$$ds^2 = a^2 R^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) - d\tau^2 \quad (2)$$

having set $\left[-\left(\frac{dt}{d\tau}\right)^2 + \left(\frac{d}{d\tau}a(t)\right)^2 R^2(t) \right] = -1$ the coefficient of $d\tau^2$. Notice that the physical radius of the wormhole is aR . Using $a' = \dot{a} \frac{dt}{d\tau}$, where the prime and the dot represent the derivatives with respect to τ and t respectively, one obtains

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{1 - (\dot{a}R)^2}} \quad (3)$$

therefore it will be possible in the following to eliminate one of the two time parameters τ and t in favour of the other. The second fundamental form (extrinsic curvature) is given by

$$K_{ij}^{\pm} = -n_{\gamma} \left(\frac{\partial^2 X^{\gamma}}{\partial \xi^i \partial \xi^j} + \Gamma_{\alpha\beta}^{\gamma\pm} \frac{\partial X^{\alpha}}{\partial \xi^i} \frac{\partial X^{\beta}}{\partial \xi^j} \right) \quad (4)$$

where n_{γ} is the unit normal to Σ :

$$n_{\gamma} = \pm \left| g^{\alpha\beta} \frac{\partial f}{\partial X^{\alpha}} \frac{\partial f}{\partial X^{\beta}} \right|^{-1/2} \frac{\partial f}{\partial X^{\gamma}} \quad (5)$$

With the definitions $\kappa_{ij} = K_{ij}^{+} - K_{ij}^{-}$ the Lanczos equations follow from the Einstein field equations for the hypersurface, and are given by

$$S_j^i = -\frac{1}{8\pi} (\kappa_j^i - \delta_j^i \kappa_h^h) \quad (6)$$

where S_j^i is the surface stress-energy tensor on Σ . In the case of spherical symmetry considerable simplifications occur, namely $\kappa_{ij} = \text{diag}(\kappa_{\vartheta}^{\vartheta}, \kappa_{\vartheta}^{\vartheta}, \kappa_{\tau}^{\tau})$, and the surface stress-energy tensor may be written in terms of the surface pressure \mathcal{P} and the surface energy density σ as $S_j^i = \text{diag}(\mathcal{P}, \mathcal{P}, -\sigma)$. The Lanczos equations then reduce to

$$\mathcal{P} = \frac{1}{8\pi} (\kappa_{\vartheta}^{\vartheta} + \kappa_{\tau}^{\tau}) \quad (7)$$

$$\sigma = -\frac{1}{4\pi} \kappa_{\vartheta}^{\vartheta} \quad (8)$$

where

$$\kappa_{\vartheta}^{\vartheta} = a' \dot{R} + \frac{\sqrt{1 + (a'R)^2}}{aR} \quad (9)$$

$$\kappa_{\tau}^{\tau} = \frac{a''R}{\sqrt{1 + (a'R)^2}} + 2a' \dot{R} \quad (10)$$

Using Eq. (3), it is possible to substitute derivatives with respect to the proper time on the shell τ with derivatives with respect to the cosmic time t , so the components of the tensor

S_j^i are writtem as

$$\mathcal{P} = \frac{[1 - (\dot{a}R)^2]^{-1} \frac{d}{dt} (\dot{a}R) + 2\dot{a}\dot{R} + \frac{1}{aR}}{4\pi \sqrt{1 - (\dot{a}R)^2}} \quad (11)$$

$$\sigma = - \frac{\dot{a}\dot{R} + \frac{1}{aR}}{2\pi \sqrt{1 - (\dot{a}R)^2}} \quad (12)$$

and the conservation identity $S_{j;i}^i = 0$, which is not independent but can be obtained from the field equations, provides the simple relationship

$$\dot{\sigma} + 2 \left(\frac{\dot{a}}{a} + \frac{\dot{R}}{R} \right) (\mathcal{P} + \sigma) = 0. \quad (13)$$

If one chooses a particular equation of state, in the form $\mathcal{P} = \mathcal{P}(\sigma)$, then the conservation equation can be formally integrated obtaining

$$\int \frac{d\sigma}{\mathcal{P}(\sigma) + \sigma} + 2 \ln(aR) = 0 \quad . \quad (14)$$

The choice $\mathcal{P} = \mathcal{P}(\sigma)$, because of Eqs. (11) and (12), will set a bond between a and R . In this paper we shall consider the case

$$\mathcal{P} = w\sigma \quad (15)$$

where w is an arbitrary real constant. Therefore the conservation equation (14) and the equation of state (15) become respectively

$$\sigma = \sigma_0 \left(\frac{a_0 R_0}{aR} \right)^{2(1+w)} \quad (16)$$

$$[1 - (\dot{a}R)^2]^{-1} \frac{d}{dt} (\dot{a}R) + 2(1+w) \dot{a}\dot{R} + \frac{(1+2w)}{aR} = 0 \quad (17)$$

where the subscript $_0$ refers to quantities calculated at some initial time $t = t_0$. We have four unknowns: a , R , \mathcal{P} and σ , but only three independent equations: (11), (12) and (15). The standard stability method based on the definition of a potential⁸ is not applicable here because from those equations it does not appear possible to obtain explicitly the derivative with respect to the time of the physical wormhole radius aR as a function of aR . Thus, to close the system, we need to have one more equation. A suitable fourth equation must be chosen in such a way to facilitate the stability analysis for the examined class of wormholes.

3. Stability analysis

In this section, to analyze the stability of the wormhole physical radius we substitute again, using Eq. (3), derivatives of aR with respect to τ with derivatives with respect to t and it may be checked that at equilibrium, where $\frac{d}{d\tau}(aR)$ and $\frac{d}{dt}(aR)$ vanish, the final result with regard to the stability does not change. We consider the wormhole radius aR as a dynamical system (see, e.g., Ref. 23) with an evolution equation of the form

$$\frac{d}{dt}(aR) = F(aR, \lambda) \quad (18)$$

where F is a differentiable function acting on the spacetime where aR is defined and λ denotes a set of control parameters affecting the evolution. The value of aR for which $F(aR, \lambda) = 0$ is a steady state point. The stability of the equilibrium will be studied considering small perturbations away from the above fixed point and determining, by means of a linear analysis, the tendency of the perturbations to grow or to decay in time. Now

$$\frac{d}{dt}(aR) = \dot{a}R + a\dot{R} \quad (19)$$

so we must know both $\dot{a}R$ and $a\dot{R}$ as a function of aR . Here it is worth noticing that when $w = -1/2$ the equation of state (17) becomes

$$[1 - (\dot{a}R)^2]^{-1} \frac{d}{dt}(\dot{a}R) + \dot{a}\dot{R} = 0 \quad (20)$$

which is satisfied either when the throat function a is constant, and so $\dot{a} = 0$, or when a is time-dependent, and \dot{a} is given integrating Eq. (20):

$$|\dot{a}| = \frac{1}{R} \left\{ 1 + \frac{[1 - (\dot{a}_0 R_0)^2]}{(\dot{a}_0 R_0)^2} \left(\frac{R}{R_0} \right)^2 \right\}^{-1/2}. \quad (21)$$

When $\dot{a} = 0$, the wormhole radius aR simply varies in direct proportion to the scale factor R . When $\dot{a} \neq 0$, which is the case treated in this paper, the behavior of aR can be studied if the scale factor R is known. When $w \neq -1/2$, to choose a suitable fourth equation we consider a specific class of simple wormhole solutions corresponding to the choice

$$[1 - (\dot{a}R)^2]^{-1} \frac{d}{dt}(\dot{a}R) - 2w \dot{a}\dot{R} = 0 \quad (22)$$

which generalizes Eq. (20) to the other values of w , and the equation of state (17) becomes

$$(1 + 2w) \left(2\dot{a}\dot{R} + \frac{1}{aR} \right) = 0 \quad (23)$$

In the following we shall consider values of w different from the value $w = -1/2$ which was briefly discussed above, so Eq. (23) gives

$$2\dot{a}\dot{R} + \frac{1}{aR} = 0 \quad . \quad (24)$$

Then we shall treat a particular class of wormholes for which \dot{a} and \dot{R} must have opposite signs and therefore when one is increasing the other is decreasing. Now we substitute $a\dot{R} = -1/(2\dot{a}R)$ into Eq. (19) which becomes

$$\frac{d}{dt}(aR) = \dot{a}R - \frac{1}{2\dot{a}R} \quad (25)$$

therefore the equilibrium is reached when $(\dot{a}R)^2 = 1/2$. Moreover the components of the tensor S_j^i now become

$$\mathcal{P} = - \frac{w}{4\pi aR\sqrt{1 - (\dot{a}R)^2}} \quad (26)$$

$$\sigma = - \frac{1}{4\pi aR\sqrt{1 - (\dot{a}R)^2}} \quad (27)$$

so the weak energy condition is violated, while the null energy condition is satisfied when $w \leq -1$. Finally, to obtain $\dot{a}R$ as a function of aR we equate Eqs. (16) and (27) for the surface density σ and have

$$1 + 4\pi a_0 R_0 \sigma_0 \sqrt{1 - (\dot{a}R)^2} \left(\frac{a_0 R_0}{aR} \right)^{1+2w} = 0 \quad . \quad (28)$$

Evaluating the previous equation at the time $t = t_0$ it results

$$1 + 4\pi a_0 R_0 \sigma_0 \sqrt{1 - (\dot{a}_0 R_0)^2} = 0 \quad . \quad (29)$$

Then Eq. (28) takes the form

$$\sqrt{\frac{1 - (\dot{a}R)^2}{1 - (\dot{a}_0 R_0)^2}} = \left(\frac{aR}{a_0 R_0} \right)^{1+2w} \quad (30)$$

and therefore

$$|\dot{a}R| = \sqrt{1 - [1 - (\dot{a}_0 R_0)^2] \left(\frac{aR}{a_0 R_0} \right)^{2(1+2w)}} \quad . \quad (31)$$

Then, when $w = 0$ both $\dot{a}R$, by Eq. (22), and aR are constant. Now we can write Eq. (25) in the form of Eq. (18):

$$\frac{d}{dt}(aR) = \text{sign}[\dot{a}] \frac{\frac{1}{2} - [1 - (\dot{a}_0 R_0)^2] \left(\frac{aR}{a_0 R_0} \right)^{2(1+2w)}}{\sqrt{1 - [1 - (\dot{a}_0 R_0)^2] \left(\frac{aR}{a_0 R_0} \right)^{2(1+2w)}}} \quad (32)$$

where $\text{sign}[\dot{a}] = \dot{a}/|\dot{a}|$. The right-hand side vanishes when

$$a_* R_* = a_0 R_0 [2 [1 - (\dot{a}_0 R_0)^2]]^{-1/(2(1+2w))} \quad (33)$$

so $a_* R_*$ is the static solution which gives the radius of the throat at the equilibrium. If $(\dot{a}_0 R_0)^2 = 1/2$, then the static solution is $a_0 R_0$ and does not depend on w . The role of perturbation of stability will be expressed by setting in Eq. (32)

$$aR = a_* R_* + \xi(t) \quad (34)$$

where $|\xi| \ll a_* R_*$ represents the disturbance. We are interested in linear analysis so higher order terms will be neglected in the Taylor expansion of the evolution equation (32) which yields at first order

$$\frac{d}{dt} \xi = -\text{sign}[\dot{a}] \sqrt{2} (1 + 2w) \frac{\xi}{a_* R_*} \quad (35)$$

Therefore we can conclude, at least until the nonlinear region is reached, that :

(i) if $(1+2w) \text{sign}[\dot{a}] > 0$, then the perturbation $\xi(t)$ decays exponentially and the equilibrium is stable; (ii) if $(1+2w) \text{sign}[\dot{a}] < 0$, then the perturbation $\xi(t)$ grows exponentially and the equilibrium is unstable. Here \dot{a} and \dot{R} have opposite signs, so in an expanding universe the equilibrium is stable when $w < -1/2$. Finally we recall that, having chosen the equation of state in the form $\mathcal{P} = \mathcal{P}(\sigma)$, the functions a and R are related. This can be seen explicitly integrating first Eq. (22):

$$|\dot{a}R| = \left\{ 1 + \frac{[1 - (\dot{a}_0 R_0)^2]}{(\dot{a}_0 R_0)^2} \left(\frac{R}{R_0} \right)^{-4w} \right\}^{-1/2} \quad (36)$$

and then equating, in the case $w \neq -1/2$, Eqs. (31) and (36). The result is

$$\frac{a}{a_0} = \frac{R_0}{R} \left\{ [1 - (\dot{a}_0 R_0)^2] + (\dot{a}_0 R_0)^2 \left(\frac{R}{R_0} \right)^{4w} \right\}^{-1/(2(1+2w))} \quad (37)$$

4. Conclusion

In this paper we have considered a particular class of thin-shell wormholes evolving in a flat FRW cosmological background and analyzed their stability to linearized perturbations around static solutions using the stability analysis of dynamical systems. The physical radius of the wormholes is equal to aR , where a is a time-dependent function describing

the dynamics of the throat and R is the background scale factor. We have considered the equation of state $\mathcal{P} = w\sigma$ and found that the factor w is crucial for determining the stability of the equilibrium. Wormholes supported by other equations of state than the one used in this paper can be used to generate other families of additional solutions.

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